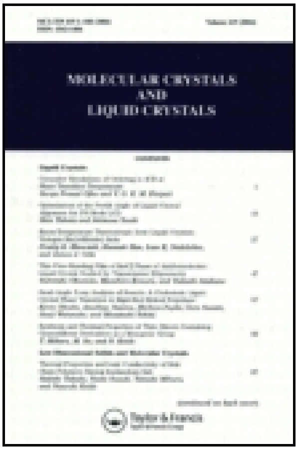

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ON SHORT SURFACE WAVES IN NEMATIC LIQUID CRYSTALS

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On Short Surface Waves in Nematic Liquid Crystals

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A question of short gravitational wave propagation on incompressible liquid crystal surface with anysotropic surface tension is discussed. Volume phase is described by Oseen model and surface phase – by Rapini models. Isotropic viscosity is taken into account. The influence of divergent member of free energy on surface stability is analysed.

Keywords: surface wave; nematic; stability

I. MAIN EQUATIONS AND BOUNDARY CONDITIONS

For nematic liquid crystal models anisotropy is described by the unit vector $\mathbf{n}$, which is called director. It may be connected with the average orientation of longer axis of molecules or more complex structures. We consider Oseen model for the elastic orientation energy of director field [1]. This model has additional symmetry group [2] which is connected with the independent director rotation. Then medium equations are significantly simplified and it is possible to solve completely the problem of small surface wave propagation.

The Oseen energy has the form

$$F_V = \frac{1}{2}K\nabla_i n_j \nabla^i n^j + \frac{1}{2}K_0\left(\nabla_i n_j \nabla^j n^i - (\nabla_k n^k)^2\right) \qquad (1)$$

The second term in (1) is the divergent addend. It does not affect on the internal orientation equations, but we need to consider it in the boundary conditions.

The surface energy is given by Rapini form

$$F_S = \alpha + \frac{\beta}{2}\left(1 - (\mathbf{n}, \mathbf{b})^2\right) \tag{2}$$

where the unit vector **b** is the light orientation axis on the surface of the nematic. The angle $\Omega \in [0, \pi/2]$ between **b** and unit normal vector **m** is constant which depends on the media interface properties. Also we suppose that α, β are constants, usually $\beta/\alpha \leq 0.1$ [4].

It is possible to show [3] that F_S has minimum when **n**, **b** and **m** are coplanar vectors. Then (2) is reduced to the expression

$$F_S = \alpha + \frac{\beta}{2}\left(1 - (\sin\Omega\sqrt{1 - n_m^2} + \cos\Omega|n_m|)^2\right)$$

where $n_m = (\mathbf{n}, \mathbf{m})$.

In this case incompressible fluid motion and director orientation equations in Cartesian coordinates under homogeneous gravity field **g** have the form

$$\nabla_i v^i = 0, \qquad \rho\frac{dv^i}{dt} = \nabla_j p^{ij} + \rho g^i \tag{3}$$

$$p^{ij} = -\nabla^i n^k \frac{\partial F_V}{\partial \nabla_j n^k} - p\,\delta^{ij} + 2\mu e^{ij}, \qquad e^{ij} = \frac{1}{2}(\nabla^i v^j + \nabla^j v^i)$$

$$\left(\delta_k^j - n^j n_k\right)\nabla_i \frac{\partial F_V}{\partial \nabla_i n^j} = 0 \tag{4}$$

where v^i are the velocity components ($i = 1, 2, 3$), p is the pressure, ρ is the medium density, δ_k^j is the Kronecker symbol and we also neglect the director inertia. According to the Oseen idea we suppose that dissipative effects are reduced to the isotropic viscosity with viscosity coefficient μ, e^{ij} is strain rate tensor. Here ρ, μ are constants.

On the free surface, besides kinematic formula $v_m = D$, where D is the normal surface velocity, we have conditions [3]

$$p_i^j m_j = \nabla_\alpha \sigma_i{}^\alpha - p_a m_i \tag{5}$$

$$\sigma^i{}_\alpha = x_\alpha^i F_S - n_\alpha m^i \frac{dF_S}{dn_m}$$

$$\left(\delta_k^j - n^j n_k\right)\left(\frac{\partial F_V}{\partial \nabla_i n^j} m_i + \frac{dF_S}{dn_m} m_j\right) = 0 \tag{6}$$

where ${\sigma^i}_\alpha$ are the surface tension tensor components, p_a is the external pressure.

Surface covariant derivative ∇^α $(\alpha = 1, 2)$ is calculated under surface metric $a_{\alpha\beta} = g_{ij} x^i_\alpha x^j_\beta$, where $x^i_\alpha = \partial x^i / \partial u^\alpha$ are the tangent vector components, u^α are the surface coordinates.

Using (4), motion equations (3) are simplified to

$$\rho \frac{dv^i}{dt} + \nabla^i (p + F_V) = \mu \Delta v^i + \rho g^i$$

So we can redefine pressure p, and orientation and motion equations are separated, and velocity and director are connected by boundary conditions only.

Due to the condition (6) relations (5) are reduced to (see [3])

$$e_{\alpha m} = 0, \tag{7}$$

$$2\mu\, e_{mm} + n^\alpha \nabla_\alpha \left(\frac{dF_S}{dn_m}\right) + \nabla_i n^i \frac{dF_S}{dn_m} = b^\alpha_\alpha \left(F_S - n_m \frac{dF_S}{dn_m}\right) + p - p_a \tag{8}$$

where $b_{\alpha\beta} = m_i \nabla_\alpha x^i_\beta$ is the tensor of surface second quadratic form.

II. SOLUTION OF THE LINEAR PROBLEM

In framework of given model we consider the small amplitude simple harmonic wave propagation problem for fluid with infinite depth. Let x, y, z be Cartesian coordinates. z–axis is directed contrariwise $\mathbf{g}$ and x–axis is along horizontal part of wave vector $\mathbf{k}$. We can write

$$\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, cos\theta)$$

Then for equilibrium state $\theta_0 = \Omega$ and φ_0 are given constant angles and $p_0 = p_a - \rho g z$.

In general case, when $\Omega > 0$, linearized equations (3), (4) for perturbation have the form

$$\nabla_i v^i = 0, \qquad \rho v^i_t + \nabla^i \tilde{p} = \mu \Delta v^i \tag{9}$$

$$\Delta\theta = 0, \qquad \Delta\varphi = 0 \tag{10}$$

Here $v^i_t = \partial v^i/\partial t$, $\tilde{p} = p - p_0$ and Δ is the Laplace operator.

We research the solution of equations (9), (10) in the form

$$v^i = \mathrm{Re}(u^i(z)\,E)), \qquad \tilde{p} = \mathrm{Re}(q(z)\,E)$$

$$\theta = \Omega + \mathrm{Re}(r(z)\,E), \qquad \varphi = \varphi_0 + \mathrm{Re}(s(z)\,E)$$

with the free surface equation $z = \mathrm{Re}(Q\,E)$, where k is a real positive number, Q is a complex constant and $E = \exp(i(\omega\,t - k\,x))$.

Using linearized boundary conditions (6), (7), (8) and perturbation damping conditions of all desired functions at $z \to -\infty$, we receive (about the calculations of velocity and pressure see, for example, [5])

$$u_1 = -i(A\exp(kz) + Bl/k\,\exp(lz)), \qquad u_2 = 0$$

$$u_3 = A\exp(kz) + B\exp(lz), \qquad q = -i\rho\,A\omega/k\,\exp(kz)$$

$$r = C\exp(kz), \qquad s = i\,\gamma\sin\varphi_0\,r(z)$$

where

$$A = -B\left(1 + \frac{i\omega\rho}{2\mu k^2}\right), \qquad l = k\sqrt{1 + \frac{i\omega\rho}{\mu k^2}}$$

$$Q = -\frac{\rho\,B}{2\mu k^2}, \qquad C = \frac{i\,k\,\beta\cos\varphi_0\,Q}{k\,K(1 - (\gamma\,\sin\Omega\,\sin\varphi_0)^2) + \beta}$$

$\gamma = K_0/K$, B is an arbitrary complex constant.

Also the following variance ratio takes place

$$\rho(\omega^2 - k\,g) - \tilde{\alpha}(k)\,k^3 = 4\mu\,k^2\left(i\,\omega + (1 - l/k)\,\mu\,k^2/\rho\right) \tag{11}$$

$$\tilde{\alpha} = \alpha + \frac{\cos\varphi_0^2\,\beta\,k\,K}{kK + \beta/(1 - (\gamma\,\sin\Omega\,\sin\varphi_0)^2)}$$

which determines the frequency value ω.

III. CASE $\Omega = 0$

In special case $\Omega = 0$ for Oseen model linearized equations and boundary conditions for p, v^i and θ are separated and solved independently as in the previous section. Variable coefficient linear equation for φ

$$\nabla_i\left(\sin^2\theta\,\nabla^i\varphi\right) = 0 \tag{12}$$

$$\theta = \mathrm{Re}\,(C\ \exp(kz)E(t,x)), \qquad C = ik\beta\cos\varphi_0\, Q/(kK+\beta)$$

is solved after using the boundary condition at $z = 0$

$$K\varphi_z + K_0 \sin\varphi_0\theta_x = 0 \tag{13}$$

and damping one at $z \to -\infty$.

In main approximation, (12) has the form ($\nu = \arg\, C$)

$$\varphi_{xx} + \varphi_{zz} + 2k\ (\tan(\omega t - kx + \nu)\,\varphi_x + \varphi_z) = 0 \tag{14}$$

We seek the solution of equation (14) as

$$\varphi = \varphi_0 + \varphi_1(z)\,\varphi_2(\xi), \qquad \xi = \omega t - kx + \nu$$

Then as result of variables separation and ordinary differential equation solution we obtain

$$\varphi_1 = \exp(\sigma z), \qquad \sigma = k\left(\sqrt{1+\lambda/k^2} - 1\right)$$

$$\varphi_2 = \frac{a\ \cos(\xi(1+\sigma/k)+b)}{\cos\xi}$$

where $\lambda > 0$, a, b are real constants.

From the condition (13) we have

$$2a\sigma K\ \cos(\xi(1+\sigma/k)+b) + kK_0|C|\sin\varphi_0\ \sin 2\xi = 0$$

From there $\sigma = k$ and finally

$$\varphi = \varphi_0 - \gamma|C|\sin\varphi_0\ \sin\xi\ \exp(kz), \qquad \theta = |C|\ \cos\xi\ \exp(kz)$$

It gives us the same formula for $s(z)$ as in the general case (see Section II). In the variance ratio (11) we are to set $\Omega = 0$.

IV. DISCUSSION OF RESULTS

The research of relation (11) shows that if $|\gamma| > 1/\sin\Omega$ hence for suitable φ_0 the value of k exists always, then $\mathrm{Im}\,\omega < 0$ and surface is unstable. So, for surface stability it is necessary that

$$|\gamma| \le \frac{1}{\sin\Omega}$$

Usually at the modelling it is supposed $\gamma = 0$, then always $s \equiv 0$ and for Ossen model any director rotation about vertical axis is absent. In case $\cos\varphi_0 = 0$, when the wave goes perpendicularly to horizontal projection of $\mathbf{n_0}$, both angle perturbations are equal to zero.

We also give the frequency asymptotics for small and large Reynolds numbers

$$\frac{|\omega|\rho}{\mu\, k^2} \to 0 : \qquad \omega \approx i\frac{\rho\, g + \tilde{\alpha}\, k^2}{2\mu\, k}$$

$$\frac{|\omega|\rho}{\mu\, k^2} \to \infty : \qquad \omega \approx 2i\mu\, k^2/\rho + \sqrt{g\, k + \tilde{\alpha}\, k^3/\rho}$$

We can emphasize that the surface tension effective coefficient $\tilde{\alpha}$ depends on the wave number k.

References

[1] P.G. de Gennes, *The physics of liquid crystals*, Clarendon Press, Oxford, 1984.
[2] J.L. Ericksen, *Issledovaniya po mekhanike sploshnykh sred*, (*Studies on continuum mechanics*), in russian, **13**, Mir, Moscow, 1977.
[3] A.G. Kalugin, A.N. Golubiatnikov, *Trudy MIRAN* (*Proc. V.A. Steklov Inst. Math.*), **223**, 1998.
[4] L.M. Blinov, E.I. Kats, A.A. Sonin, *Uspekhi Phys. Nauk*, **152**, no. 3, 1987.
[5] L.D. Landau, E.M. Lifshits. *Fluid mechanics*. Pergamon Press. Oxford; New York, 3 Edition, 1983.